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Some properties of the restricted NCP-functions for the nonlinear complementarity problem

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Abstract. In this paper, we study restricted NCP-functions which may be used to reformulate the nonlinear complementarity problem as a constrained minimization problem. In particular, we consider three classes of restricted NCP-functions, two of which were introduced by Solodov and the other is proposed in this paper. We give conditions under which a minimization problem based on a restricted NCP-function enjoys favorable properties, such as the equivalence between a stationary point of the minimization problem and the nonlinear complementarity problem, the strict complementarity at a solution of the minimization problem and the boundedness of level sets of the objective function of the minimization problem. We examine those properties for the three restricted NCP-functions and show that the merit function constituted by the restricted NCP-function proposed in this paper enjoys quite favorable properties compared with those based on the other restricted NCP-functions.

Key words: Nonlinear complementarity problem, restricted NCP-function, merit function, constrained optimization reformulation, bounded level sets.

1. Introduction

The nonlinear complementarity problem (NCP) [3] is to find a vector $x \in R^n$ such that

$$\text{[NCP]} \qquad \langle x, F(x)\rangle = 0,\ x \geq 0,\ F(x) \geq 0,$$

where F is a continuously differentiable mapping from R^n into itself and $\langle\cdot,\cdot\rangle$ denotes the inner product in R^n.

One of the popular approaches to solve the NCP is to reformulate the original NCP as a minimization problem whose global minima are coincident with the solutions of the NCP [4, 7, 8, 9, 10, 12, 13, 14, 17]. The objective function of such an equivalent minimization problem is called a merit function for the NCP. Most of the merit functions considered in the above references are constituted using an NCP-function [4, 9] $\phi : R^2 \to R$, which satisfies the property

$$\phi(a,b) = 0 \Longleftrightarrow ab = 0, a \geq 0, b \geq 0.$$

The implicit Lagrangian [13], the squared Fischer-Burmeister function [9], and the class of functions which are considered in [10] are such merit functions that are constituted by NCP-functions.

Recently, these functions have drawn much attention and have been shown to enjoy many favorable properties [4, 7, 8, 9, 10, 12, 13, 14, 17].

In this paper, we are particularly interested in the case where the function F involved in the NCP is defined only on the nonnegative orthant $R^n_+ := \{x \in R^n \mid x \geq 0\}$. As pointed out in [2], we often encounter such NCP in some applications. For such NCP, it is natural that a merit function f is also defined only on R^n_+. Thus we are led to the nonnegatively constrained minimization problem:

$$\min_{x \geq 0} \ f(x). \tag{1}$$

Facchinei and Kanzow [2] considered the constrained minimization problem (1) based on the implicit Lagrangian. Fischer [5] studied the problem (1) based on the squared Fischer-Burmeister function. In particular, they gave necessary and sufficient conditions under which any Karush-Kuhn-Tucker (KKT) point of the problem (1), i.e., a point x satisfying

$$\lambda = \nabla f(x) \ \geq \ 0, \tag{2}$$
$$x \ \geq \ 0, \tag{3}$$
$$\langle x, \nabla f(x) \rangle \ = \ 0, \tag{4}$$

is a solution of the NCP. However, since the implicit Lagrangian and the squared Fischer-Burmeister function are such merit functions that can constitute an equivalent unconstrained optimization problem for the NCP, an optimal Lagrange multiplier λ of the problem (1) must satisfy

$$\lambda = \nabla f(x) = 0.$$

Hence the strict complementarity, i.e,

$$x_i + \lambda_i > 0 \quad \text{for all } i,$$

does not hold in general. There exist a number of methods that converge rapidly to a solution of a constrained minimization problem without the strictly complementarity property [11, 14]. Nevertheless, the strict complementarity still turns out to be important in theoretical analysis of optimization algorithms. Therefore, it is natural to look for a merit function which yields a nonnegatively constrained minimization problem whose KKT point satisfies the strict complementarity under reasonable conditions. For constructing such merit functions, it is convenient to consider the class of restricted NCP-functions defined as follows.

Definition 1.1 *Let $S = \{(a,b)^T \in R^2 \mid a \geq 0\}$. A function $\phi : S \to R$ is called a restricted NCP-function if, for $(a,b)^T \in S$,*

$$\phi(a,b) = 0 \iff b \geq 0, ab = 0.$$

We call ϕ a nonnegative restricted NCP-function if ϕ is a restricted NCP-function and $\phi(a,b) \geq 0$ for all $(a,b)^T \in S$.

Note that Solodov [15] has also considered merit functions constituted by nonnegative restricted NCP-functions and has shown some favorable properties of those merit functions. However, the strict complementarity of the problem (1) is not discussed in [15].

In this paper, we consider the merit function f defined by

$$f(x) = \sum_{i=1}^{n} \phi(x_i, F_i(x)), \tag{5}$$

where ϕ is a nonnegative restricted NCP-function. The following theorem says that the function f constitutes a nonnegatively constrained minimization problem equivalent to the NCP. The proof is evident, hence is omitted.

Theorem 1.1 *Let f be defined by (5) with a nonnegative restricted NCP-function ϕ. Then an $x \in R^n$ satisfies $x \geq 0$ and $f(x) = 0$ if and only if x solves the NCP.* □

The following three functions are restricted NCP-functions, as shown in Theorem 1.2 below.

(i) $\phi_{RG}(a,b) = ab + \frac{1}{2\alpha}([a - \alpha b]_+^2 - a^2)$, $\alpha > 0$,

(ii) $\phi_S(a,b) = a[b]_+^2 + [-b]_+^2$,

(iii) $\phi_A(a,b) = a[b]_+^3 + \frac{1}{2}(\sqrt{a^2+b^2} - a - b)^2$.

The function ϕ_{RG} constitutes the regularized gap function proposed by Fukushima [6]. The function ϕ_S was recently proposed by Solodov [15]. Solodov [15] gives necessary and sufficient conditions under which a KKT point of the problem (1) based on ϕ_{RG} or ϕ_S is a solution of the NCP. On the other hand, the function ϕ_A is the augmented squared Fischer-Burmeister function with the additional term $a[b]_+^3$. Because of this additional term, ϕ_A enjoys a number of favorable properties, as shown later.

Remark. The term $a[b]_+^3$ that appear in the definition of ϕ_A may be replaced with $a[b]_+^\gamma$ where $\gamma > 1$, without affecting the desirable properties of ϕ_A that will be shown later. For simplicity, however, we restrict ourselves to the case where $\gamma = 3$.

The functions ϕ_{RG}, ϕ_S and ϕ_A are differentiable and their derivatives are given by

$$\begin{pmatrix} \frac{\partial \phi_{RG}(a,b)}{\partial a} \\ \frac{\partial \phi_{RG}(a,b)}{\partial b} \end{pmatrix} = \begin{pmatrix} b - \frac{1}{\alpha}a + \frac{1}{\alpha}[a - \alpha b]_+ \\ a - [a - \alpha b]_+ \end{pmatrix}, \tag{6}$$

$$\begin{pmatrix} \frac{\partial \phi_S(a,b)}{\partial a} \\ \frac{\partial \phi_S(a,b)}{\partial b} \end{pmatrix} = \begin{pmatrix} [b]_+^2 \\ 2a[b]_+ - 2[-b]_+ \end{pmatrix}, \tag{7}$$

$$\begin{pmatrix} \frac{\partial \phi_A(a,b)}{\partial a} \\ \frac{\partial \phi_A(a,b)}{\partial b} \end{pmatrix} = \begin{cases} \begin{pmatrix} [b]_+^3 + \left(\frac{a}{\sqrt{a^2+b^2}} - 1\right)(\sqrt{a^2+b^2} - a - b) \\ 3a[b]_+^2 + \left(\frac{b}{\sqrt{a^2+b^2}} - 1\right)(\sqrt{a^2+b^2} - a - b) \end{pmatrix} & \text{if } (a,b) \neq (0,0) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } (a,b) = (0,0). \end{cases} \tag{8}$$

Note that $(\frac{\partial \phi_A(a,b)}{\partial a}, \frac{\partial \phi_A(a,b)}{\partial b})$ is continuous at the origin. Moreover ϕ_A is twice differentiable at any point $(a,b) \neq (0,0)$, while ϕ_{RG} is twice differentiable at any point (a,b) such that $a \geq 0, b \geq 0, ab = 0$ and $(a,b) \neq (0,0)$.

The next theorem shows that the functions ϕ_{RG}, ϕ_S and ϕ_A are nonnegative restricted NCP-functions.

Theorem 1.2 *The functions ϕ_{RG}, ϕ_S and ϕ_A are nonnegative restricted NCP-functions.*

Proof. That the functions ϕ_{RG} and ϕ_S are restricted NCP-functions is proven in [15]. So, we only consider the function ϕ_A. Let $a \geq 0$. First, suppose that $b \geq 0$ and $ab = 0$. Then, since $\sqrt{a^2+b^2}-a-b$ is an NCP-function, we have

$$\phi_A(a,b) = 0.$$

Conversely, suppose that $\phi_A(a,b) = 0$. Then, since the first and second terms of ϕ_A are nonnegative, the second term must be zero, i.e., $\sqrt{a^2+b^2}-a-b = 0$. It follows from the fact that $\sqrt{a^2+b^2}-a-b$ is an NCP-function that $a \geq 0, b \geq 0$ and $ab = 0$. The proof is complete. □

Theorems 1.1 and 1.2 show that the NCP is equivalent to the nonnegatively constrained problem (1) with the objective function f constituted by any of the functions ϕ_{RG}, ϕ_S or ϕ_A.

The purpose of the paper is to investigate conditions under which the problem (1) defined by a restricted NCP-function has favorable properties. In particular, we study conditions under which any KKT point of the problem (1) becomes a solution of the NCP. Note that such conditions are given by Solodov [15] for the restricted NCP-functions ϕ_{RG} and ϕ_S. In addition to a result similar to [15], we will show that any KKT point of the problem (1) defined by the restricted NCP-function ϕ_A is a solution of the NCP under weaker conditions than those given in [15]. Moreover we give conditions under which the strict complementarity holds at a KKT point and conditions under which the merit function f has bounded level sets. These conditions appear to be new as far as the nonnegatively constrained minimization reformulation of the NCP is concerned. In particular, we show that the new restricted NCP-function ϕ_A enjoys all of these favorable properties.

We use the following notation. For an index set $I \subseteq \{1,2,\cdots,n\}$ and an n-dimensional vector x, x_I stands for the vector consisting of the components of x whose indices are in I. The vector e_i denotes the ith column vector of the identity matrix. For a function $\phi : R^2 \to R$, $\frac{\partial\phi(x,F(x))}{\partial a}$ and $\frac{\partial\phi(x,F(x))}{\partial b}$ denote the vectors $(\frac{\partial\phi(x_1,F_1(x))}{\partial a},\cdots,\frac{\partial\phi(x_n,F_n(x))}{\partial a})^T$ and $(\frac{\partial\phi(x_1,F_1(x))}{\partial b},\cdots,\frac{\partial\phi(x_n,F_n(x))}{\partial b})^T$, respectively. We define the set Q by $Q := \{(a,b)^T \in R^2 \mid a \geq 0, b \geq 0, ab = 0\}$. For a solution x of the NCP and the index sets $J(x) := \{i \mid x_i = 0\}$ and $K(x) := \{i \mid F_i(x) = 0\}$, we say that the solution x is nondegenerate if the set $J(x) \cap K(x)$ is empty, and x is degenerate, otherwise.

2. KKT points of the minimization problem

In this section, we give necessary and sufficient conditions for a KKT point of the problem (1) based on a restricted NCP-function to be a solution of the NCP. For this purpose, we define special classes of restricted NCP-functions.

Definition 2.1 *Φ_+ denotes the class of restricted NCP-functions ϕ such that*

C.1 *$a \geq 0$ and $\frac{\partial\phi(a,b)}{\partial b} = 0$ if and only if $(a,b)^T \in Q$;*

C.2 *$\frac{\partial\phi(a,b)}{\partial a}\frac{\partial\phi(a,b)}{\partial b} \geq 0$ for all $a \geq 0$ and $b \in R$;*

C.3 *$\frac{\partial\phi(0,b)}{\partial b} \leq 0$ for all $b \in R$.*

Moreover, Φ_{++} denotes the class of functions $\phi \in \Phi_+$ such that

C.4 $a \geq 0$ *and* $\frac{\partial \phi(a,b)}{\partial a} \frac{\partial \phi(a,b)}{\partial b} = 0 \Rightarrow (a,b)^T \in Q$.

Note that C.1 in the definition says that the function $\frac{\partial \phi}{\partial b}$ is also a restricted NCP-function.

Next, we show that both ϕ_{RG} and ϕ_S belong to Φ_+, and ϕ_A belongs to Φ_{++}.

Theorem 2.1 *The functions ϕ_{RG} and ϕ_S belong to Φ_+. The function ϕ_A belongs to Φ_{++}.*

Proof. The first half follows directly from [15, Lemma 2.3]. So we only show $\phi_A \in \Phi_{++}$. First, note that

$$(a,b)^T \in Q \Longleftrightarrow \sqrt{a^2+b^2} - a - b = 0. \tag{9}$$

(C.1) Suppose that $(a,b)^T \in Q$. It is clear that $a \geq 0$. Moreover, by $ab = 0$ and (9), the equality

$$\frac{\partial \phi_A(a,b)}{\partial b} = 0$$

follows from (8).

Conversely, suppose that $a \geq 0$ and $\frac{\partial \phi_A(a,b)}{\partial b} = 0$. If $(a,b) = (0,0)$, then $(a,b)^T \in Q$. Now assume that $(a,b) \neq (0,0)$. Since $a \geq 0$, the inequality $a[b]_+^2 \geq 0$ always holds. We consider two cases $a[b]_+^2 = 0$ and $a[b]_+^2 > 0$. Let $a[b]_+^2 = 0$. Then we have

$$\frac{\partial \phi_A(a,b)}{\partial b} = \left(\frac{b}{\sqrt{a^2+b^2}} - 1\right)(\sqrt{a^2+b^2} - a - b) = 0,$$

which implies that $\frac{b}{\sqrt{a^2+b^2}} - 1 = 0$ or $\sqrt{a^2+b^2} - a - b = 0$. If $\frac{b}{\sqrt{a^2+b^2}} - 1 = 0$, then we have $a = 0$ and $b > 0$, which implies $(a,b)^T \in Q$. If $\sqrt{a^2+b^2} - a - b = 0$, we also have $(a,b)^T \in Q$ by (9). Next, let $a[b]_+^2 > 0$. Then, we have $a > 0$ and $b > 0$, and hence

$$\left(\frac{b}{\sqrt{a^2+b^2}} - 1\right)(\sqrt{a^2+b^2} - a - b) > 0.$$

Since this contradicts $\frac{\partial \phi_A(a,b)}{\partial b} = 0$, the strict inequality $a[b]_+^2 > 0$ does not hold when $a \geq 0$ and $\frac{\partial \phi_A(a,b)}{\partial b} = 0$. Consequently, the converse is also true.

(C.2) Since the statement is clear when $(a,b) = (0,0)$, we suppose that $(a,b) \neq (0,0)$. First, note that the inequalities

$$\frac{a}{\sqrt{a^2+b^2}} - 1 \leq 0 \quad \text{and} \quad \frac{b}{\sqrt{a^2+b^2}} - 1 \leq 0 \tag{10}$$

always hold. When $a \geq 0$, the following inequality holds:

$$[b]_+(\sqrt{a^2+b^2} - a - b) \leq 0 \quad \text{for any } b. \tag{11}$$

It follows from (8) that, for any $a \geq 0$ and b,

$$\begin{aligned}
\frac{\partial \phi_A(a,b)}{\partial a} \frac{\partial \phi_A(a,b)}{\partial b} &= 3a[b]_+^5 + 3a[b]_+^2 \left(\frac{a}{\sqrt{a^2+b^2}} - 1\right)(\sqrt{a^2+b^2} - a - b) \\
&\quad + [b]_+^3 \left(\frac{b}{\sqrt{a^2+b^2}} - 1\right)(\sqrt{a^2+b^2} - a - b) \\
&\quad + \left(\frac{a}{\sqrt{a^2+b^2}} - 1\right)\left(\frac{b}{\sqrt{a^2+b^2}} - 1\right)(\sqrt{a^2+b^2} - a - b)^2 \\
&\geq 0,
\end{aligned}$$

where the last inequality follows from (10) and (11).

(C.3) The case where $b = 0$ is evident. Suppose that $b \neq 0$. Then, by (8) we have

$$\begin{aligned}\frac{\partial \phi_A(0,b)}{\partial b} &= \left(\frac{b}{\sqrt{b^2}} - 1\right)(\sqrt{b^2} - b) \\ &\leq 0.\end{aligned}$$

(C.4) $\frac{\partial \phi(a,b)}{\partial a}\frac{\partial \phi(a,b)}{\partial b} = 0$ implies $\frac{\partial \phi(a,b)}{\partial a} = 0$ or $\frac{\partial \phi(a,b)}{\partial b} = 0$. If $a \geq 0$ and $\frac{\partial \phi_A(a,b)}{\partial b} = 0$, then we have $(a,b)^T \in Q$ by (C.1). Next, we consider the case where $a \geq 0$ and $\frac{\partial \phi_A(a,b)}{\partial a} = 0$. If $(a,b) = (0,0)$, then $(a,b)^T \in Q$. Let $(a,b) \neq (0,0)$. If $[b]_+ = 0$, by (8), we have

$$\frac{\partial \phi_A(a,b)}{\partial a} = \left(\frac{a}{\sqrt{a^2+b^2}} - 1\right)(\sqrt{a^2+b^2} - a - b) = 0.$$

Hence we can prove $(a,b)^T \in Q$ in a way similar to the proof of (C.1). Moreover, we can also show that $[b]_+ > 0$ does not hold, in a way similar to the proof of (C.1). Consequently, we obtain the desired relation. $\square$

Note that the functions ϕ_{RG} and ϕ_S do not satisfy (C.4). In fact, consider the case where $a = \alpha b > 0$ for ϕ_{RG} and the case where $a \geq 0$ and $b < 0$ for ϕ_S. Note that $(a,b)^T \notin Q$ in these cases. However, $\frac{\partial \phi_{RG}(a,b)}{\partial a} = 0$ when $a = \alpha b > 0$, and $\frac{\partial \phi_S(a,b)}{\partial a} = 0$ when $a \geq 0$ and $b < 0$.

Now we proceed to give the main result of the section. To this end, we define the index sets

$$\begin{aligned}\mathcal{P}(x) &= \left\{i \,\middle|\, \frac{\partial \phi(x_i, F_i(x))}{\partial b} > 0\right\}, \\ \mathcal{C}(x) &= \left\{i \,\middle|\, \frac{\partial \phi(x_i, F_i(x))}{\partial b} = 0\right\}, \\ \mathcal{N}(x) &= \left\{i \,\middle|\, \frac{\partial \phi(x_i, F_i(x))}{\partial b} < 0\right\}.\end{aligned}$$

When the point x under consideration is clear from the context, we shall denote these sets simply by $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{N}$.

Using these index sets, we define regularity conditions, which are slight modifications of those given in [10]. (See also [1, 2, 5, 14, 15] for similar definitions.)

Definition 2.2 *A point $x \in R^n_+$ is said to be regular if for every $z \in R^n (z \neq 0)$ such that*

$$z_{\mathcal{C}} = 0, \quad z_{\mathcal{P}} > 0, \quad z_{\mathcal{N}} < 0, \tag{12}$$

there exists a vector $y \in R^n$ such that

$$y_{\mathcal{C}} = 0, \quad y_{\mathcal{P}} \geq 0, \quad y_{\mathcal{N}} \leq 0, \quad y_{\mathcal{P}\cup\mathcal{N}} \neq 0 \tag{13}$$

and

$$\langle y, \nabla F(x) z\rangle \geq 0. \tag{14}$$

Moreover, a point $x \in R^n_+$ is strictly regular if for every $z \in R^n (z \neq 0)$ such that

$$z_{\mathcal{C}} = 0, \quad z_{\mathcal{P}} > 0, \quad z_{\mathcal{N}} < 0,$$

there exists a vector $y \in R^n$ such that

$$y_{\mathcal{C}} = 0, \;\; y_{\mathcal{P}} \geq 0, \;\; y_{\mathcal{N}} \leq 0$$

and

$$\langle y, \nabla F(x)z \rangle > 0.$$

Note that a regular point is strictly regular. The definition of regularity contains the condition $y_{\mathcal{C}} = 0$ in (13), which is different from that in [10]. However, in a way similar to [10, Lemma 5.2], we can show that x is regular in the sense of Definition 2.2 if $\nabla F(x)$ is a P_0-matrix. Moreover, x is strictly regular if $\nabla F(x)$ is a P-matrix [2].

By using these definitions, we state the main result of the section.

Theorem 2.2 *Let x be a KKT point of the problem (1) defined by a restricted NCP-function ϕ. Suppose that $\phi \in \Phi_+$. Then x is a solution of the NCP if and only if x is strictly regular. Moreover, suppose that $\phi \in \Phi_{++}$. Then x is a solution of the NCP if and only if x is regular.*

Proof. We can prove the cases $\phi \in \Phi_+$ and $\phi \in \Phi_{++}$ by slightly modifying the corresponding proofs in [2] and [5], respectively. Here, we only show the last part of the theorem.

If x is a solution of the NCP, then $\frac{\partial\phi(x,F(x))}{\partial b} = 0$ by condition (C.1) in Definition 2.1. Hence, there is no vector z satisfying (12), which implies that x is regular.

Conversely, suppose that a point $x \geq 0$ is regular. Assume that x is not a solution of the NCP. Let $z = \frac{\partial\phi(x,F(x))}{\partial b}$. Then $z \neq 0$ by (C.1) and z satisfies (12). Moreover, by Definition 2.2, there exists a vector y satisfying (13) and (14). Let $I := \{i \mid [\nabla f(x)]_i > 0\}$. Then, since $x_I = 0$ by the KKT conditions (2)-(4), we have $z_I \leq 0$ by (C.3). It follows from (13) that $y_I \leq 0$. Hence, we have

$$\begin{aligned} \langle y, \nabla f(x) \rangle &= \langle y_I, [\nabla f(x)]_I \rangle \\ &\leq 0. \end{aligned} \tag{15}$$

On the other hand, we have

$$\begin{aligned} \nabla f(x) &= \frac{\partial\phi(x,F(x))}{\partial a} + \nabla F(x)\frac{\partial\phi(x,F(x))}{\partial b} \\ &= \frac{\partial\phi(x,F(x))}{\partial a} + \nabla F(x)z. \end{aligned}$$

It follows from (15) that

$$\langle y, \nabla f(x) \rangle = \langle y, \frac{\partial\phi(x,F(x))}{\partial a} \rangle + \langle y, \nabla F(x)z \rangle \leq 0. \tag{16}$$

Since $z_i\frac{\partial\phi(x_i,F_i(x))}{\partial a} \geq 0$ for all i by (C.2) and since both z_i and y_i have the same sign by (13), the inequality $y_i\frac{\partial\phi(x_i,F_i(x))}{\partial a} \geq 0$ holds for all i. Now, suppose that $y_i\frac{\partial\phi(x_i,F_i(x))}{\partial a} = 0$ for every i. If $\frac{\partial\phi(x_i,F_i(x))}{\partial a} = 0$, then $(x_i, F_i(x))^T \in Q$ by (C.4). Thus, we have $i \in \mathcal{C}$, and hence $y_i = 0$. For the case where $\frac{\partial\phi(x_i,F_i(x))}{\partial a} \neq 0$, we also have $y_i = 0$. Consequently we have $y = 0$, which contradicts (13). Hence, there exists i such that $y_i\frac{\partial\phi(x_i,F_i(x))}{\partial a} > 0$, which implies $\langle y, \frac{\partial\phi(x,F(x))}{\partial a} \rangle > 0$. It then follows from (16) that $\langle y, \nabla f(x)z\rangle < 0$, which contradicts (14). This means that x is a solution of the NCP. □

Remark. There exists a counterexample showing that a KKT point x of the merit functions constituted by ϕ_{RG} and ϕ_S is not a solution of the NCP even if x is regular. In fact, consider the simple example with $n = 1$, where $F(x) = (x-1)^3 - 1$ for all $x \in R$. Note that F is strictly monotone, and hence x is regular. Let $f_{RG}(x) = \phi_{RG}(x, F(x))$ and $f_S(x) = \phi_S(x, F(x))$. If $\alpha = 1$, then we have $\nabla f_{RG}(x, F(x)) = 0$ when $x = 1$. Also we have $\nabla f_S(x, F(x)) = 0$ when $x = 1$. However, $x = 1$ is not a solution of the NCP.

3. The strict complementarity and the second-order optimality conditions

In this section, we give a condition under which the strict complementarity holds at a KKT point of the problem (1). We also consider the second-order optimality condition for the problem (1).

The following theorem relates a nondegenerate solution of the NCP with the strict complementarity in the problem (1).

Theorem 3.1 *Let $\phi \in \Phi_+$. Suppose that $\frac{\partial\phi(a,b)}{\partial a} = 0$ when $a > 0$ and $b = 0$ and that $\frac{\partial\phi(a,b)}{\partial a} > 0$ when $a = 0$ and $b > 0$. If x is a nondegenerate solution of the NCP, then x is a KKT point of the problem (1) which satisfies the strict complementarity condition.*

Proof. It is evident that a solution of the NCP satisfies the KKT conditions (2)-(4). Hence, we only consider the strict complementarity. Let x be a nondegenerate solution of the NCP. Then by (C.1) in Definition 2.1, we have $\frac{\partial\phi(x,F(x))}{\partial b} = 0$. It follows from (2) that an optimal Lagrange multiplier of the problem (1) is given by

$$\lambda = \nabla f(x) = \frac{\partial\phi(x,F(x))}{\partial a} + \nabla F(x)\frac{\partial\phi(x,F(x))}{\partial b} = \frac{\partial\phi(x,F(x))}{\partial a}.$$

By assumption, when $x_i > 0$ and $F_i(x) = 0$, $\lambda_i = \frac{\partial f(x)}{\partial x_i} = 0$ holds, and when $x_i = 0$ and $F_i(x) > 0$, $\lambda_i = \frac{\partial f(x)}{\partial x_i} > 0$ holds. Since x is a nondegenerate solution of the NCP, the strict complementarity

$$x_i + \lambda_i > 0$$

holds for all i. □

The following theorem says that the functions ϕ_{RG}, ϕ_S and ϕ_A satisfy the assumption of the above theorem.

Theorem 3.2 *The functions ϕ_{RG}, ϕ_S and ϕ_A satisfy the assumption of Theorem 3.1.*

Proof. Let $a > 0$ and $b = 0$. Then by (6), (7) and (8), we have

$$\begin{aligned}
\frac{\partial\phi_{RG}(a,0)}{\partial a} &= -\frac{1}{\alpha}a + \frac{1}{\alpha}[a]_+ = 0, \\
\frac{\partial\phi_S(a,0)}{\partial a} &= 0, \\
\frac{\partial\phi_A(a,0)}{\partial a} &= \left(\frac{a}{\sqrt{a^2}} - 1\right)(\sqrt{a^2} - a) = 0.
\end{aligned}$$

Next, let $a = 0$ and $b > 0$. Then by (6), (7) and (8), we have

$$\begin{aligned}
\frac{\partial \phi_{RG}(0,b)}{\partial a} &= b + \frac{1}{\alpha}[-\alpha b]_+ = b > 0, \\
\frac{\partial \phi_S(0,b)}{\partial a} &= [b]_+^2 > 0, \\
\frac{\partial \phi_A(0,b)}{\partial a} &= [b]_+^3 - (\sqrt{b^2} - b) = [b]_+^3 > 0.
\end{aligned}$$

□

The next corollary follows directly from the above two theorems.

Corollary 3.1 *Let the problem (1) be defined by either of ϕ_{RG}, ϕ_S and ϕ_A. Then, a nondegenerate solution x of the NCP is a KKT point of the problem (1) satisfying the strict complementarity condition.* □

In the remainder of the section, we consider the second-order optimality conditions for the problem (1) defined by ϕ_{RG} and ϕ_A. Let f_{RG} and f_A denote the merit functions constituted from ϕ_{RG} and ϕ_A, respectively.

Using the two well-known functions, the implicit Lagrangian [13] and the squared Fischer-Burmeister function [9], we prove the main theorem of this section. For this purpose, we consider two functions which relate f_{RG} and f_A to the implicit Lagrangian and the squared Fischer-Burmeister function, respectively.

Lemma 3.1 *Let $p, q : R^n \to R$ be given by*

$$\begin{aligned}
p(x) &= \sum_{i=1}^{n} x_i [F_i(x)]_+^3, \\
q(x) &= \|F(x)\|^2 - \|[F(x) - \alpha x]_+\|^2.
\end{aligned}$$

The function p is everywhere twice differentiable and q is twice differentiable at nondegenerate solutions of the NCP. Moreover, the functions p and q are nonnegative on R^n_+ and the following inequalities hold at a nondegenerate solution x of the NCP:

$$\begin{aligned}
\langle y, \nabla^2 p(x) y \rangle &\geq 0 \quad \textit{for all } y \in T(x), \\
\langle y, \nabla^2 q(x) y \rangle &\geq 0 \quad \textit{for all } y \in T(x),
\end{aligned}$$

where $T(x)$ is given by

$$T(x) := \{y \in R^n \mid y_i = 0, i \in J(x)\}.$$

Proof. By definition, we can easily show that p is nonnegative on R^n_+. To show the nonnegativity of q on R^n_+, it is sufficient to prove that $b^2 - [b - \alpha a]_+^2$ is nonnegative for any $a \geq 0$ and b. If $b - \alpha a \geq 0$, then

$$\begin{aligned}
b^2 - [b - \alpha a]_+^2 &= 2\alpha ab - \alpha^2 a^2 \\
&= 2\alpha a(b - \alpha a) + \alpha^2 a^2 \\
&\geq 0.
\end{aligned}$$

If $b - \alpha a < 0$, then $b^2 - [b - \alpha a]_+^2 = b^2$, which is nonnegative. Hence, q is nonnegative on R_+^n. The differentiability of p and q follows directly from the definition. Moreover, since $p(x) = q(x) = 0$ at a solution x of the NCP, p and q attain their minimum on R_+^n at x. Hence, the latter part of the lemma follows from the second-order necessary optimality conditions for the problems

$$\min_{x \geq 0} p(x)$$

and

$$\min_{x \geq 0} q(x).$$

□

Now by using the above lemma, we give the desired condition.

Theorem 3.3 *Suppose $\alpha > 1$ in ϕ_{RG}. Let x be a nondegenerate solution such that $\nabla F_i(x)(i \in K(x))$ and $e_i(i \in J(x))$ are linearly independent. Then the following inequalities hold:*

$$\begin{aligned} \langle y, \nabla^2 f_{RG}(x) y \rangle &> 0 \quad \text{for all } y \in T(x), y \neq 0, \\ \langle y, \nabla^2 f_A(x) y \rangle &> 0 \quad \text{for all } y \in T(x), y \neq 0. \end{aligned}$$

Proof. Let M_α be the implicit Lagrangian [13] and let f_{FB} be the squared Fischer-Burmeister function [9]. Then, under the given assumptions, $\nabla^2 M_\alpha(x)$ and $\nabla^2 f_{FB}(x)$ exist and are positive definite matrices [9, 13]. Since f_{RG} and f_A can be expressed as

$$\begin{aligned} f_{RG}(x) &= M_\alpha(x) + \frac{1}{2\alpha} q(x), \\ f_A(x) &= f_{FB}(x) + p(x), \end{aligned}$$

respectively, the desired inequalities follow from Lemma 3.1. □

Note that the assumption that x is nondegenerate is implicitly contained in the linear independence condition, since otherwise there would at least be $n + 1$ linearly independent vectors in R^n.

4. Bounded level sets

In this section, we consider conditions under which a merit function f defined by a restricted NCP-function ϕ has bounded level sets. Such a property guarantees that the sequence produced by a descent method applied to the problem (1) has an accumulation point.

The following result appears to be new as far as the constrained minimization reformulation of the NCP are concerned. The proof is similar to [10, Theorem 4.1], which is concerned with the unconstrained minimization reformulations of the NCP.

Theorem 4.1 *Let ϕ be a restricted NCP-function such that*

$$b \to -\infty \text{ or } ab \to \infty \Longrightarrow \phi(a, b) \to \infty.$$

Suppose that F is monotone and the NCP is strictly feasible, i.e., there exists a vector $\hat{x} > 0$ with $F(\hat{x}) > 0$. Then the level sets $\mathcal{L}(c) := \{x \in R_+^n \mid f(x) \leq c\}$ are bounded for all $c \geq 0$.

Proof. Assume that there exists an unbounded sequence $\{x^k\} \subseteq \mathcal{L}(c)$ for some $c \geq 0$. Since the sequence $\{x_i^k\}$ is nonnegative for all i, there exists an index j such that $x_j^k \to \infty$ on a subsequence. Let $\hat{x}$ be a strictly feasible point of the NCP. Since F is monotone, we have

$$\langle x^k, F(\hat{x})\rangle + \langle \hat{x}, F(x^k)\rangle \leq \langle x^k, F(x^k)\rangle + \langle \hat{x}, F(\hat{x})\rangle. \tag{17}$$

Since $b \to -\infty$ implies $\phi(a,b) \to \infty$, $\{F_i(x^k)\}$ must be bounded below for every i. It follows from $\hat{x} > 0$ and $F(\hat{x}) > 0$ that the left-hand side of (17) tends to infinity on a subsequence. Hence, $\langle x^k, F(x^k)\rangle \to \infty$. Thus, there exists an index i such that $x_i^k F_i(x^k) \to \infty$ on a subsequence. Since $\phi(x_i^k, F_i(x^k)) \to \infty$ by assumption, we have $f(x^k) \to \infty$. This contradicts $\{x^k\} \subseteq \mathcal{L}(c)$. The proof is complete. □

Next, we show that the functions ϕ_S and ϕ_A satisfy the assumption of Theorem 4.1.

Theorem 4.2 *The restricted NCP-functions ϕ_S and ϕ_A satisfy the assumption of Theorem 4.1.*

Proof. Each term of ϕ_S and ϕ_A is nonnegative when $a \geq 0$. For either of ϕ_S and ϕ_A, the first term tends to infinity if $a \geq 0$ and $ab \to \infty$, while the second term tends to infinity if $a \geq 0$ and $b \to -\infty$. Hence, both functions ϕ_S and ϕ_A satisfy the assumption of Theorem 4.1. □

Note that ϕ_{RG} does not satisfy the assumption of Theorem 4.1. In fact, consider the case where $a > \alpha b > 0$ and b is constant. Then we have

$$\begin{aligned}\phi_{RG}(a,b) &= ab + \frac{1}{2\alpha}(a^2 - 2\alpha ab + \alpha^2 b^2 - a^2) \\ &= \frac{1}{2}\alpha b^2.\end{aligned}$$

Thus, $a \to \infty$ implies $ab \to \infty$, but does not imply $\phi_{RG}(a,b) \to \infty$. Moreover, the following counterexample shows that level sets of the regularized gap function may not be bounded even if F is monotone and the NCP is strictly feasible. Let $n = 1$, $F(x) \equiv 1$ and $\alpha = 1$. Then F is monotone and the NCP is strictly feasible. However, we have

$$\mathcal{L}(1) := \left\{ x \in R_+ \;\middle|\; xF(x) + \frac{1}{2}([x - F(x)]_+^2 - x^2) \leq 1 \right\} = [0, \infty),$$

which is unbounded. Note that this example also serves as a counterexample showing that level sets of the squared Fischer-Burmeister function and the implicit Lagrangian function are not necessarily bounded under the same conditions.

The next theorem gives another results on the boundedness of level sets.

Theorem 4.3 *Suppose that F is a uniform P-function. Then the levels sets of the merit function constituted by either ϕ_{RG} or ϕ_A are bounded.*

Proof. It is well known that the squared Fischer-Burmeister function and the natural residual function have bounded level sets when F is a uniform P-function [8, 16]. By the definitions of ϕ_{RG} and ϕ_A, the merit functions constituted by ϕ_{RG} and ϕ_A dominate the squared Fischer-Burmeister function and the natural residual function multiplied by a positive constant, respectively. Hence, the level sets of the merit functions constituted by ϕ_A and ϕ_{RG} are also bounded if F is a uniform P-function. □

Table 1: Various merit functions for the NCP

	KKT point	s.c.	level sets	twice diff.
implicit Lagrangian	strictly regular	no	no	yes
squared F-B	regular	no	no	yes
regularized gap	strictly regular	yes	no	yes
Solodov's function	strictly regular	yes	yes	no
the proposed function	regular	yes	yes	yes

"KKT point" denotes a condition for a KKT point to be a solution of the NCP, "s.c." denotes the strict complementarity at a KKT point of the problem (1), "level sets" denotes the boundness of the level sets of the merit function under the monotonicity of F and strict feasibility of the NCP, and "twice diff." denotes the twice differentiability at a nondegenerate solution.

5. Concluding remarks

In this paper, we have investigated conditions under which the problem (1) defined by a restricted NCP-function has favorable properties and examined those properties for three particular restricted NCP-functions. In Table 1, we summarize the properties of the implicit Lagrangian, the squared Fischer-Burmeister function (squared F-B) and the merit functions constituted by the three restricted NCP-functions considered in the paper. The table shows that the merit function constituted by the restricted NCP-function ϕ_A proposed in this paper enjoys quite favorable properties compared with other merit functions.

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